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LETTER TO THE EDITOR

The classical capacity of a quantum dense coding system

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Abstract
Quantum dense coding transmits classical information by sending a quantum system with the assistance of quantum entanglement. The classical information capacity of a quantum dense coding system is obtained, where a sender and receiver share a completely entangled state and a quantum system encoded by applying unitary operators is sent through an arbitrary quantum channel. The result is compared with that obtained in another setting.

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Quantum dense coding is a method for transmitting classical information by sending an encoded quantum system with the assistance of quantum entanglement [1]. It was originally proposed for ideal systems in which Alice (a sender) and Bob (a receiver) share a completely entangled state and a quantum channel is noiseless [1–3], and it was then generalized for systems which use mixed entangled states and noisy quantum channels [4, 5]. If one does not use quantum entanglement, the upper bound of the classical information capacity is $\log_2 N$ bits when a quantum system which carries the information is described by an N-dimensional Hilbert space $\mathcal{H}$. Quantum entanglement, however, can enhance the classical information capacity greater than $\log_2 N$ bits. In quantum dense coding, Alice and Bob share an arbitrary bipartite quantum state $\hat{\rho}^{AB}$ defined on an $(N \times N)$-dimensional Hilbert space $\mathcal{H} \otimes \mathcal{H}$. To encode $2\log_2 N$ bits of classical information, Alice applies one of N^2 unitary operators to her part of the quantum state $\hat{\rho}^{AB}$ and sends the encoded system to Bob through a quantum channel. After receiving it, Bob performs a quantum measurement to extract the information encoded by Alice. If the quantum channel is noiseless, where the output state is identical with the input state, one can obtain the classical information capacity $C(\hat{\rho})$ of the quantum dense coding system [6, 7], which is given by

$$C(\hat{\rho}) = \log_2 N + S(\hat{\rho}^{B}) - S(\hat{\rho}^{AB}) \quad (1)$$

where $\hat{\rho}^{\rm B}$ is the reduced quantum state obtained by $\hat{\rho}^{\rm B} = {\rm Tr}_{\rm A}\,\hat{\rho}^{\rm AB}$ and $S(\hat{\rho})$ is the von Neumann entropy of a quantum state $\hat{\rho}$, namely, $S(\hat{\rho}) = -{\rm Tr}[\hat{\rho}\log_2\hat{\rho}]$. For a completely entangled state $\hat{\rho}^{\rm AB}$ which yields $S(\hat{\rho}^{\rm AB}) = 0$ and $S(\hat{\rho}^{\rm B}) = \log_2 N$, the capacity $C(\hat{\rho}) = 2\log_2 N$ is obtained.

This letter derives the classical information capacity $C(\hat{\mathcal{L}})$ of a quantum dense coding system, where Alice and Bob share a completely entangled state $|\Psi^{\rm AB}\rangle$ which belongs to an $(N\times N)$-dimensional Hilbert space $\mathcal{H}\otimes\mathcal{H}$ and the encoded system is sent through a quantum channel described by an arbitrary trace-preserving completely positive map $\hat{\mathcal{L}}$. The capacity $C(\hat{\mathcal{L}})$ is compared with $C(\hat{\rho})$ given by equation (1).

A completely entangled state $|\Psi^{\rm AB}\rangle$ of an $(N\times N)$-dimensional Hilbert space $\mathcal{H}\otimes\mathcal{H}$ shared by Alice and Bob can be written as

$$|\Psi^{\rm AB}\rangle = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}\left|\psi_k^{\rm A}\right\rangle\otimes\left|\psi_k^{\rm B}\right\rangle \tag{2}$$

where $\{|\psi_k\rangle | k = 0, 1, \ldots, N-1\}$ is a complete orthonormal system of the N-dimensional Hilbert space $\mathcal{H}$. One introduces N^2 unitary operators defined on the N-dimensional Hilbert space $\mathcal{H}$, which are called the unitary depolarizers [1, 7]

$$\hat{U}_{jk} = \sum_{l=0}^{N-1}{\rm e}^{(2\pi{\rm i}/N)jl}|\psi_{l\,{\rm mod}\,N}\rangle\langle\psi_{l+k\,{\rm mod}\,N}| \tag{3}$$

with $j, k = 0, 1, \ldots, N-1$. It is an easy task to see that the unitary depolarizers satisfy the relations

$$\frac{1}{N}\sum_{j=0}^{N-1}\sum_{k=0}^{N-1}\hat{U}_{jk}\hat{X}\hat{U}_{jk}^{\dagger} = ({\rm Tr}\,\hat{X})\hat{1} \qquad \frac{1}{N}{\rm Tr}\left[\hat{U}_{jk}\hat{U}_{lm}^{\dagger}\right] = \delta_{jl}\delta_{km} \tag{4}$$

for any operator $\hat{X}$ defined on the Hilbert space $\mathcal{H}$. These relations imply that the set of completely entangled states, $\left\{\left|\Psi_{jk}^{\rm AB}\right\rangle = \left(\hat{U}_{jk}^{\rm A}\otimes\hat{1}^{\rm B}\right)|\Psi^{\rm AB}\rangle\middle| j, k = 0, 1, \ldots, N-1\right\}$, becomes a complete orthonormal system of the Hilbert space $\mathcal{H}\otimes\mathcal{H}$.

We suppose that to perform the quantum dense coding, Alice and Bob share a statistical mixture of the completely entangled states $\left|\Psi_{jk}^{\rm AB}\right\rangle$, where the shared quantum state $\hat{\rho}^{\rm AB}$ is given by

$$\hat{\rho}^{\rm AB} = \sum_{j=0}^{N-1}\sum_{k=0}^{N-1}\lambda_{jk}\left|\Psi_{jk}^{\rm AB}\right\rangle\left\langle\Psi_{jk}^{\rm AB}\right| \tag{5}$$

with $\lambda_{jk}\geqslant 0$ and $\sum_{j=0}^{N-1}\sum_{k=0}^{N-1}\lambda_{jk} = 1$. To encode $2\log_2 N$ bits of classical information, Alice applies one of N^2 unitary operators $\hat{V}_{jk}$ $(j, k = 0, 1, \ldots, N-1)$ to her part of the quantum state $\hat{\rho}^{\rm AB}$ and sends the encoded system to Bob through an arbitrary quantum channel $\hat{\mathcal{L}}$. Then Bob obtains the quantum state

$$\hat{\rho}_{jk}^{\rm AB\prime} = (\hat{\mathcal{L}}^{\rm A}\otimes\hat{\mathcal{I}}^{\rm B})\left[\left(\hat{V}_{jk}^{\rm A}\otimes\hat{1}^{\rm B}\right)\hat{\rho}^{\rm AB}\left(\hat{V}_{jk}^{{\rm A}\dagger}\otimes\hat{1}\right)\right]. \tag{6}$$

When the prior probability of the classical information 'jk' is π_{jk}, where $\pi_{jk}\geqslant 0$ and $\sum_{j=0}^{N-1}\sum_{k=0}^{N-1}\pi_{jk} = 1$, the Holevo function $\chi(\hat{\mathcal{L}};\{\pi_{jk}\})$ [8] of the quantum dense coding system is given by

$$\chi(\hat{\mathcal{L}};\{\pi_{jk}\}) = S(\hat{\rho}^{\rm AB\prime}) - \sum_{j=0}^{N-1}\sum_{k=0}^{N-1}\pi_{jk}S\left(\hat{\rho}_{jk}^{\rm AB\prime}\right) \tag{7}$$

where $\hat{\rho}^{\mathrm{AB}\prime} = \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \pi_{jk} \hat{\rho}_{jk}^{\mathrm{AB}\prime}$. The classical information capacity $C(\hat{\mathcal{L}})$ of the system is the maximum value of the Holevo function $\chi(\hat{\mathcal{L}}; \{\pi_{jk}\})$ with respect to the prior probability π_{jk}, that is,

$$C(\hat{\mathcal{L}}) = \max_{\{\pi_{jk}\}} \chi(\hat{\mathcal{L}}; \{\pi_{jk}\}). \tag{8}$$

To obtain the classical information capacity $C(\hat{\mathcal{L}})$, we first estimate the Holevo function $\chi(\hat{\mathcal{L}}; \{\pi_{jk}\})$ given by equation (7). Since the completely entangled state $|\Psi_{jk}^{\mathrm{AB}}\rangle$ satisfies the relation $(\hat{X}^{\mathrm{A}} \otimes \hat{1}^{\mathrm{B}})|\Psi_{jk}^{\mathrm{AB}}\rangle = (\hat{1}^{\mathrm{A}} \otimes \hat{X}^{\mathrm{BT}})|\Psi_{jk}^{\mathrm{AB}}\rangle$ for any operator $\hat{X}$ defined on the N-dimensional Hilbert space $\mathcal{H}$, where the symbol 'T' stands for the transposition of operators, and the von Neumann entropy $S(\hat{\rho})$ is invariant under any unitary transformation of a quantum state $\hat{\rho}$, the second term on the right-hand side of equation (7) becomes

$$\sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \pi_{jk} S\big(\hat{\rho}_{jk}^{\mathrm{AB}\prime}\big) = S((\hat{\mathcal{L}}^{\mathrm{A}} \otimes \hat{\mathcal{I}}^{\mathrm{B}})\hat{\rho}^{\mathrm{AB}}). \tag{9}$$

On the other hand, since we have the relation $\mathrm{Tr}_{\mathrm{B}}\, \hat{\rho}^{\mathrm{AB}} = (1/N)\hat{1}^{\mathrm{A}}$ and $\mathrm{Tr}_{\mathrm{A}}\, \hat{\rho}^{\mathrm{AB}} = (1/N)\hat{1}^{\mathrm{B}}$ from equation (5), the sub-additivity of the von Neumann entropy yields the inequality

$$S(\hat{\rho}^{\mathrm{AB}\prime}) \leqslant \log_2 N + S((1/N)\hat{\mathcal{L}}^{\mathrm{A}}\hat{1}^{\mathrm{A}}). \tag{10}$$

Hence from equation (7), the Holevo function $\chi(\hat{\mathcal{L}}; \{\pi_{jk}\})$ satisfies the inequality

$$\chi(\hat{\mathcal{L}}; \{\pi_{jk}\}) \leqslant \log_2 N + S((1/N)\hat{\mathcal{L}}^{\mathrm{A}}\hat{1}^{\mathrm{A}}) - S((\hat{\mathcal{L}}^{\mathrm{A}} \otimes \hat{\mathcal{I}}^{\mathrm{B}})\hat{\rho}^{\mathrm{AB}}). \tag{11}$$

Since the right-hand side of this inequality does not depend on the prior probability π_{jk} of the classical information, we obtain the inequality for the classical information capacity $C(\hat{\mathcal{L}})$ of the quantum dense coding system

$$C(\hat{\mathcal{L}}) \leqslant \log_2 N + S((1/N)\hat{\mathcal{L}}^{\mathrm{A}}\hat{1}^{\mathrm{A}}) - S((\hat{\mathcal{L}}^{\mathrm{A}} \otimes \hat{\mathcal{I}}^{\mathrm{B}})\hat{\rho}^{\mathrm{AB}}). \tag{12}$$

Next, we show that the equality of equation (12) can be attained by setting $\hat{V}_{jk} = \hat{U}_{jk}$ and $\pi_{jk} = 1/N^2\ (j, k = 0, 1, \ldots, N-1)$. In this case, using equation (4), we can calculate the quantum state $\hat{\rho}^{\mathrm{AB}\prime}$ as follows:

$$\begin{aligned}
\hat{\rho}^{\mathrm{AB}\prime} &= \frac{1}{N^2} \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} (\hat{\mathcal{L}}^{\mathrm{A}} \otimes \hat{\mathcal{I}}^{\mathrm{B}})\big[\big(\hat{U}_{jk}^{\mathrm{A}} \otimes \hat{1}^{\mathrm{B}}\big)\hat{\rho}^{\mathrm{AB}}\big(\hat{U}_{jk}^{A\dagger} \otimes \hat{1}^{\mathrm{B}}\big)\big] \\
&= (\hat{\mathcal{L}}^{\mathrm{A}} \otimes \hat{\mathcal{I}}^{\mathrm{B}}) \left[\frac{1}{N^2} \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \big(\hat{U}_{jk}^{\mathrm{A}} \otimes \hat{1}^{\mathrm{B}}\big)\hat{\rho}^{\mathrm{AB}}\big(\hat{U}_{jk}^{A\dagger} \otimes \hat{1}^{\mathrm{B}}\big) \right] \\
&= (\hat{\mathcal{L}}^{\mathrm{A}} \otimes \hat{\mathcal{I}}^{\mathrm{B}}) \left(\frac{1}{N} \hat{1}^{\mathrm{A}} \otimes \mathrm{Tr}_{\mathrm{A}}\, \hat{\rho}^{\mathrm{AB}} \right) \\
&= \frac{1}{N} \hat{\mathcal{L}}^{\mathrm{A}}\hat{1}^{\mathrm{A}} \otimes \frac{1}{N} \hat{1}^{\mathrm{B}}
\end{aligned} \tag{13}$$

from which we obtain the equality $S(\hat{\rho}^{\mathrm{AB}\prime}) = \log_2 N + S((1/N)\hat{\mathcal{L}}^{\mathrm{A}}\hat{1}^{\mathrm{A}})$. This result implies that the equality of equation (12) is attained. Therefore, it has been shown that the classical information capacity $C(\hat{\mathcal{L}})$ of the quantum dense coding system is given by

$$C(\hat{\mathcal{L}}) = \log_2 N + S((1/N)\hat{\mathcal{L}}^{\mathrm{A}}\hat{1}^{\mathrm{A}}) - S((\hat{\mathcal{L}}^{\mathrm{A}} \otimes \hat{\mathcal{I}}^{\mathrm{B}})\hat{\rho}^{\mathrm{AB}}). \tag{14}$$

In particular, when Alice and Bob share the completely entangled state $|\Psi^{\mathrm{AB}}\rangle$, the capacity $C(\hat{\mathcal{L}})$ becomes

$$C(\hat{\mathcal{L}}) = \log_2 N + S((1/N)\hat{\mathcal{L}}^{\mathrm{A}}\hat{1}^{\mathrm{A}}) - S((\hat{\mathcal{L}}^{\mathrm{A}} \otimes \hat{\mathcal{I}}^{\mathrm{B}})|\Psi^{\mathrm{AB}}\rangle\langle\Psi^{\mathrm{AB}}|). \tag{15}$$

This is the main result of this letter. If the quantum channel is noiseless ($\hat{\mathcal{L}} = \hat{\mathcal{I}}$), $C(\hat{\mathcal{L}}) = 2\log_2 N$ is obtained.

To compare the classical information capacity given by equation (15) with that given by equation (1), we note the relation between bipartite quantum states and quantum channels [9]. For a given quantum channel $\hat{\mathcal{L}}$, we can assign a bipartite quantum $\hat{\rho}_{\mathcal{L}}^{\mathrm{AB}}$ by the relation $\hat{\rho}_{\mathcal{L}}^{\mathrm{AB}} = (\hat{\mathcal{L}}^{\mathrm{A}} \otimes \hat{\mathcal{I}}^{\mathrm{B}})|\Psi^{\mathrm{AB}}\rangle\langle\Psi^{\mathrm{AB}}|$, where $|\Psi^{\mathrm{AB}}\rangle$ is a completely entangled state. On the other hand, for a given bipartite quantum state $\hat{\rho}^{\mathrm{AB}}$ that satisfies the relation $\mathrm{Tr}_{\mathrm{A}}\,\hat{\rho}^{\mathrm{AB}} = (1/N)\hat{1}^{\mathrm{B}}$, we can uniquely assign a quantum channel $\hat{\mathcal{L}}_\rho$ by the relation $\hat{\rho}^{\mathrm{AB}} = \big(\hat{\mathcal{L}}_\rho^{\mathrm{A}} \otimes \hat{\mathcal{I}}^{\mathrm{B}}\big)|\Psi^{\mathrm{AB}}\rangle\langle\Psi^{\mathrm{AB}}|$ [9]. Then we can rewrite the classical information capacity $C(\hat{\mathcal{L}})$ given by equation (15) as

$$C(\hat{\mathcal{L}}) = \log_2 N + S\big(\hat{\rho}_{\mathcal{L}}^{\mathrm{A}}\big) - S\big(\hat{\rho}_{\mathcal{L}}^{\mathrm{AB}}\big) = C(\hat{\rho}_{\mathcal{L}}) \tag{16}$$

where $C(\hat{\rho})$ is given by equation (1). On the other hand, the classical information capacity $C(\hat{\rho})$ can be expressed as

$$C(\hat{\rho}) = \log_2 N + S\big((1/N)\hat{\mathcal{L}}_\rho^{\mathrm{A}}\hat{1}^{\mathrm{A}}\big) - S\big(\big(\hat{\mathcal{L}}_\rho^{\mathrm{A}} \otimes \hat{\mathcal{I}}^{\mathrm{B}}\big)|\Psi^{\mathrm{AB}}\rangle\langle\Psi^{\mathrm{AB}}|\big) = C(\hat{\mathcal{L}}_\rho) \tag{17}$$

where we have exchanged A and B in equation (1). As an example, we consider the Werner state $\hat{\rho}_p^{\mathrm{AB}} = (1-p)|\Psi^{\mathrm{AB}}\rangle\langle\Psi^{\mathrm{AB}}| + (p/N^2)\hat{1}^{\mathrm{A}} \otimes \hat{1}^{\mathrm{B}}$ and the depolarizing channel $\hat{\mathcal{L}}_p$ defined by $\hat{\mathcal{L}}_p\hat{X} = (1-p)\hat{X} + (p/N)\hat{1}$. It is obvious that the equality $\rho_p^{\mathrm{AB}} = \big(\hat{\mathcal{L}}_p^{\mathrm{A}} \otimes \hat{\mathcal{I}}^{\mathrm{B}}\big)|\Psi^{\mathrm{AB}}\rangle\langle\Psi^{\mathrm{AB}}|$ is satisfied. Then the equality $C(\hat{\rho}_p) = C(\hat{\mathcal{L}}_p)$ holds. In fact, we obtain $C(\hat{\rho}_p) = C(\hat{\mathcal{L}}_p) \equiv C_p$, where

$$C_p = \log_2[N^2 - (N^2-1)p] + \frac{N^2-1}{N^2}p\log_2\left[\frac{p}{N^2-(N^2-1)p}\right]. \tag{18}$$

This result means that the quantum dense coding system in which the completely entangled state is shared and the encoded system is sent through the depolarizing channel yields the classical information capacity equal to that of the quantum dense coding system in which the Werner state is shared and the encoded system is sent through the noiseless quantum channel.

We finally consider the relation between the classical information capacity and the von Neumann mutual information in the quantum dense coding system. When Alice and Bob share the completely entangled state $|\Psi^{\mathrm{AB}}\rangle$, the quantum state $\hat{\rho}_{\mathrm{in}}^{\mathrm{AB}}$ of the total system just before Alice inputs the encoded system to the quantum channel is given by

$$\hat{\rho}_{\mathrm{in}}^{\mathrm{AB}} = \sum_{j=0}^{N-1}\sum_{k=0}^{N-1}\pi_{jk}\big|\Psi_{jk}^{\mathrm{AB}}\big\rangle\big\langle\Psi_{jk}^{\mathrm{AB}}\big|. \tag{19}$$

After Alice has transmitted the encoded system, the quantum state $\hat{\rho}_{\mathrm{out}}^{\mathrm{AB}}$ of the total system, which is the output system of the quantum channel, becomes

$$\hat{\rho}_{\mathrm{out}}^{\mathrm{AB}} = \sum_{j=0}^{N-1}\sum_{k=0}^{N-1}\pi_{jk}(\hat{\mathcal{L}}^{\mathrm{A}} \otimes \hat{\mathcal{I}}^{\mathrm{B}})\big|\Psi_{jk}^{\mathrm{AB}}\big\rangle\big\langle\Psi_{jk}^{\mathrm{AB}}\big|. \tag{20}$$

Then one can introduce the compound state $\hat{\rho}_{\mathrm{in-out}}^{\mathrm{AB}}$ of the input and output systems by [10, 11]

$$\hat{\rho}_{\mathrm{in-out}}^{\mathrm{AB}} = \sum_{j=0}^{N-1}\sum_{k=0}^{N-1}\pi_{jk}\big|\Psi_{jk}^{\mathrm{AB}}\big\rangle\big\langle\Psi_{jk}^{\mathrm{AB}}\big| \otimes (\hat{\mathcal{L}}^{\mathrm{A}} \otimes \hat{\mathcal{I}}^{\mathrm{B}})\big|\Psi_{jk}^{\mathrm{AB}}\big\rangle\big\langle\Psi_{jk}^{\mathrm{AB}}\big| \tag{21}$$

the reduced quantum states of which are $\hat{\rho}_{\mathrm{in}}^{\mathrm{AB}} = \mathrm{Tr}_{\mathrm{out}}\,\hat{\rho}_{\mathrm{in-out}}^{\mathrm{AB}}$ and $\hat{\rho}_{\mathrm{out}}^{\mathrm{AB}} = \mathrm{Tr}_{\mathrm{in}}\,\hat{\rho}_{\mathrm{in-out}}^{\mathrm{AB}}$. Then the von Neumann mutual information $I_{\mathrm{N}}(\mathrm{in} : \mathrm{out})$ between the input and output systems is given by

$$I_{\mathrm{N}}(\mathrm{in}:\mathrm{out}) = S\big(\hat{\rho}^{\mathrm{AB}}_{\mathrm{in}}\big) + S\big(\hat{\rho}^{\mathrm{AB}}_{\mathrm{out}}\big) - S\big(\hat{\rho}^{\mathrm{AB}}_{\mathrm{in-out}}\big). \tag{22}$$

In particular, when $\pi_{jk} = 1/N^2\,(j,k = 0,1,\ldots,N-1)$, it is easy to see from equations (19)–(21) that

$$S\big(\hat{\rho}^{\mathrm{AB}}_{\mathrm{in}}\big) = 2\log_2 N \tag{23}$$

$$S\big(\hat{\rho}^{\mathrm{AB}}_{\mathrm{out}}\big) = \log_2 N + S((1/N)\hat{\mathcal{L}}^{\mathrm{A}}\hat{1}^{\mathrm{A}}) \tag{24}$$

$$S\big(\hat{\rho}^{\mathrm{AB}}_{\mathrm{in-out}}\big) = 2\log_2 N + S((\hat{\mathcal{L}}^{\mathrm{A}}\otimes\hat{\mathcal{I}}^{\mathrm{B}}|\Psi^{\mathrm{AB}}\rangle\langle\Psi^{\mathrm{AB}}|). \tag{25}$$

Hence, we obtain the von Neumann mutual information

$$I_{\mathrm{N}}(\mathrm{in}:\mathrm{out}) = \log_2 N + S((1/N)\hat{\mathcal{L}}^{\mathrm{A}}\hat{1}^{\mathrm{A}}) - S((\hat{\mathcal{L}}^{\mathrm{A}}\otimes\hat{\mathcal{I}}^{\mathrm{B}})|\Psi^{\mathrm{AB}}\rangle\langle\Psi^{\mathrm{AB}}|) \tag{26}$$

which is equal to the classical information capacity $C(\hat{\mathcal{L}})$ given by equation (15). Therefore, when we define the compound state of the input and output systems by equation (21), we obtain the equality

$$I_{\mathrm{N}}(\mathrm{in}:\mathrm{out}) = C(\hat{\mathcal{L}}). \tag{27}$$

The result may be reasonable since the compound state given by equation (21) is the separable state of the input and output systems and thus it contains only their classical correlation.

In summary, we have derived the classical information capacity $C(\hat{\mathcal{L}})$ of the quantum dense coding system, where Alice and Bob share a completely entangled state and the encoded system is sent through an arbitrary quantum channel. The classical information capacities of the quantum dense coding system that are known up to now are summarized as follows:

	Noiseless channel	Noisy channel
Completely entangled state	$2\log_2 N$	$C(\hat{\mathcal{L}})$
Arbitrary bipartite state	$C(\hat{\rho})$	Unknown

where $C(\hat{\rho})$ and $C(\hat{\mathcal{L}})$ are given by equations (1) and (15). The capacity in the most general setting is unknown, where Alice and Bob share a given bipartite state and the encoded system is sent through an arbitrary quantum channel. In the quantum dense coding, the quantum entanglement shared by Alice and Bob is given. When the quantum entanglement shared by Alice and Bob is unlimited, the classical information capacity, called the entanglement-assisted classical capacity, is obtained [12, 13]. Furthermore, we have discussed the relation between the classical information capacity and the von Neumann mutual information of the separable compound state.

References

[1] Bennett C H and Wiesner S J 1992 *Phys. Rev. Lett.* **69** 2881
[2] Barenco A and Ekert A K 1995 *J. Mod. Opt.* **42** 1253
[3] Hausladen P, Jozsa R, Schumacher B, Westmoreland M and Wootters K W 1996 *Phys. Rev.* A **54** 1869
[4] Bose S, Plenio M B and Vedral V 2000 *J. Mod. Opt.* **47** 291
[5] Bennett C H, Shor P W, Smolin J A and Thapliyal A V 1999 *Phys. Rev. Lett.* **83** 3081
[6] Bowen G 2001 *Phys. Rev.* A **63** 022302
[7] Ban M 2002 *J. Phys. A: Math. Gen.* **35** L193

[8] Holevo A S 1973 *Probl. Inf. Transm.* **9** 177
[9] Horodecki M, Horodecki P and Horodecki R 1999 *Phys. Rev.* A **60** 1888
[10] Ohya M 1983 *IEEE Trans. Inf. Theory* **IT-29** 770
[11] Ahlswede R and Löber 2001 *IEEE Trans. Inf. Theory* **IT-47** 474
[12] Bennett C H, Shor P W, Smolin J A and Thapliyal A V 2001 *IEEE Trans. Inf. Theory* **IT-48** 2637
[13] Holevo A S 2002 *J. Math. Phys.* **43** 4326